Secondary Instabilities and Spatiotemporal Chaos in Parametric Surface Waves

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Abstract

A two dimensional model is introduced to study pattern formation, secondary instabilities and the transition to spatiotemporal chaos (weak turbulence) in parametric surface waves. The stability of a periodic standing wave state above onset is studied against Eckhaus, zig-zag and transverse amplitude modulations (TAM) as a function of the control parameter ε and the detuning. A mechanism leading to a finite threshold for the TAM instability is identified. Numerical solutions of the model are in agreement with the stability diagram, and also reveal the existence of a transition to spatiotemporal chaotic states at a finite ε. Power spectra of temporal fluctuations in the chaotic state are broadband, decaying as a power law of the frequency ω^{-z} with $z \approx 4.0$.

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A layer of incompressible fluid that is driven by a sinusoidal force normal to the free surface at rest exhibits parametrically excited surface waves, also known as Faraday waves [1]. We introduce a two dimensional model that possesses many of the features observed in experimental studies of Faraday waves: a primary instability to a standing wave pattern, a secondary instability (transverse amplitude modulation or TAM) at larger amplitudes of the dimensionless driving force ε, and a chaotic state at yet larger values of ε. Our study focuses on the large aspect ratio limit in which the wavelength of the wave is much smaller than the lateral dimension of the fluid layer.

Transitions to spatiotemporal chaos have been observed in a variety of physical systems, with recent efforts concentrating on the study of fluid systems [2]. The fact that physical properties of fluids are normally well characterized, and that in some cases the transition to chaos occurs at reasonably small values of the control parameter, makes fluid systems especially attractive for an accurate quantitative study of the chaotic state. Among these recent studies we quote rotating convection in a circular geometry [3], spiral defect chaos in Rayleigh-Bénard convection [4], and Faraday waves [5]. Theoretical work, on the other hand, has concentrated on the study of coupled lattice maps [6–8], in which simple maps that are known to be chaotic are arranged on a lattice and their dynamics coupled. Complementary studies focus on phenomenological models such as the complex Ginzburg-Landau equation [2]. This equation models the low frequency behavior of a class of systems close to onset of instability, and exhibits various types of chaotic behavior. A third approach, which we follow here, concerns the so called order parameter equations. The best known example of the latter is the Swift-Hohenberg equation used to model Rayleigh-Bénard convection in a Boussinesq fluid [9]. Its associated amplitude equation, the real Ginzburg-Landau equation, coincides with the amplitude equation directly derived from the equations governing fluid motion. The Swift-Hohenberg equation, supplemented with non-gradient terms, has been recently used to study spiral defect chaos in Rayleigh-Bénard convection [10].

Recent experiments on Faraday waves in large aspect ratio systems have revealed a number of interesting phenomena [5,11–15]. Among them, periodic standing wave patterns

near onset are found to be unstable against a TAM at some finite supercriticality [5,11,13]. Associated with this instability, temporal fluctuations of the pattern have been observed, with a characteristic time scale that decreases continuously with increasing ε. Beyond a certain value of ε, the wave patterns appear temporally chaotic and spatially disordered [5,14]. In the disordered regime, the amplitudes of a range of Fourier modes appear to exhibit Gaussian statistics [5,16] (deviations from Gaussianity have also been recently reported [14]). Interestingly, disordered wave patterns in the chaotic regime have been found to exhibit highly ordered time averages [15]. The origin of the TAM instability and its relationship with the disordered state was first studied theoretically by Ezerskii et al. [11], who derived a one dimensional model to describe the modulation. Later, Milner [17] derived a set of coupled amplitude equations for a two dimensional surface, including nonlinear interaction and damping of the waves. He showed that a pattern of square symmetry is realized near onset in fluids of low viscosity, and that it can become unstable against a TAM. However, its finite threshold and its dependence on the detuning parameter are not yet well understood [2]. Studies of chaotic wave patterns were also conducted by Rabinovich and coworkers [18], who numerically studied an amplitude equation for a pair of counter-propagating waves, and found features similar to the chaotic states observed in the experiments.

The model that we have derived for a complex order parameter ψ preserves the rotational symmetry of the fluid, reduces to Milner's amplitude equations when ψ is assumed to be modulated along particular directions of propagation, and to Ezerskii et al. one dimensional equation in the case they considered. We confine ourselves in this paper to the results obtained when only the simplest nonlinear term is retained in the model. With this choice of nonlinearity we find a standing roll wave pattern close to onset. The exact form of the nonlinear term in momentum space, and the results corresponding to different choices of nonlinearity that yield standing waves of square symmetry at onset will be presented elsewhere [19]. We then obtain the stability boundaries of the roll pattern for either phase (Eckhaus and zig-zag) or amplitude (TAM) instabilities. A small but finite nonlinear damping coefficient appears to be essential for the existence of the finite driving threshold

for the TAM instability. Unfortunately, and similarly to previous studies, we do not have a self-consistent procedure to introduce nonlinear damping into the model, probably reflecting the fact that at this level it has to be regarded as phenomenological. Numerical integration is finally used to study pattern selection above onset, and a transition to spatiotemporal chaotic states mediated by a TAM at finite ε.

The model equation that we study in this paper is,

$$\partial_t \psi = -\gamma\psi + if\psi^* + \frac{3i}{4}(1+\nabla^2)\psi + (i-\gamma_n)|\psi|^2\psi, \tag{1}$$

where $\psi(x, y, t)$ is a two-dimensional complex field. Its real and imaginary parts are linear combinations of the surface displacement away from planarity and the surface velocity potential of the irrotational part of the flow [19,20]. Equation (1) has been made dimensionless by choosing $1/q_0$ as a unit of length, and $2/\Omega$ as a unit of time (Ω is the angular frequency of the sinusoidal driving force, and q_0 is critical wavenumber at onset, determined by both Ω and the surface tension of the fluid). The linear damping coefficient is $\gamma = 4\nu q_0^2/\Omega$, where ν is the kinematic viscosity of the fluid. The nonlinear damping coefficient γ_n is of the order of γ, but its exact value is undetermined. The quantity $\varepsilon = (f-\gamma)/\gamma$ is the distance away from threshold of the primary instability, where f is the dimensionless amplitude of the driving force. The phase $e^{-i\Omega t/2}$ in the motion of the surface has been scaled out of the field $\psi(x, y, t)$. Equation (1) can be derived perturbatively from the fluid equations in the limit of weak viscous dissipation and small wave-steepness [19]. With the nonlinear term used in Eq. (1), standing roll wave patterns are observed just above $\varepsilon = 0$. We believe, however, that our findings about secondary instabilities, and the general features of the transition to spatiotemporal chaos are qualitatively similar to the more realistic case in which the standing wave pattern near onset exhibits square symmetry.

For $\varepsilon > 0$, the quiescent solution $\psi = 0$ loses stability. A new stationary roll solution can be found approximately by considering a one-mode Galerkin approximation $\psi_0(x, q) =$

$\alpha_q e^{i\theta_q}\cos(qx) + \mathcal{O}(\alpha_q^3)$ with,

$$\alpha_q^2 = \frac{q^2 - 1 - 4\gamma\gamma_n/3 + \sqrt{(q^2-1-4\gamma\gamma_n/3)^2 + (1+\gamma_n^2)\left[16(f^2-\gamma^2)/9 - (q^2-1)^2\right]}}{1+\gamma_n^2},$$

$\sin 2\theta_q = (\gamma + 3\gamma_n\alpha_q^2/4)/f$, and $\cos 2\theta_q = \frac{3}{4}(q^2 - 1 - \alpha_q^2)/f$. Another solution, which does not exist for $q = 1$ and is unstable with respect to uniform amplitude perturbations, will not be considered here. The neutral stability curve is $\varepsilon_0(q) = \sqrt{1 + \left[3(1-q^2)/(4\gamma)\right]^2} - 1$. The roll solution breaks the translational invariance of the base state. Hence a phase diffusion equation can be derived to study the stability of the roll solution. We find that the Eckhaus stability boundary is given by [21],

$$\begin{aligned} 3(\alpha_q^2 + 1 - q^2)\left(\frac{2}{3}\alpha_q^2 + 1 - q^2\right)\left(q^2 + \frac{\alpha_q^2}{2}\right) &= 2\gamma_n q^2\alpha_q^2(2\gamma + \gamma_n q^2) \\ &\quad - \gamma_n q^2\left(2\gamma + \frac{3}{2}\gamma_n q^2\right)\left(\frac{2}{3}\alpha_q^2 + 1 - q^2\right), \qquad (2) \end{aligned}$$

and the zig-zag stable region is given by $\alpha_q^2 > \frac{3}{2}(q^2-1)$. Fig. 1 shows the Eckhaus and zig-zag stability boundaries for $\gamma = 0.1$ (this value of γ is motivated by recent experiments [5,11,13]) and $\gamma_n = 0.05$. The reentrant shape of the Eckhaus boundary is a direct consequence of the existence of a small nonlinear damping coefficient γ_n. In the small ε limit ($\varepsilon \ll \gamma_n^2$), the region of stability is symmetric around $q = 1$ and is given by, $\varepsilon > \frac{27}{32\gamma^2}(1-q^2)^2$, whereas for $\gamma_n = 0$, the stable region is $\varepsilon < \sqrt{\left[1 + 3(1-q^2)/(8\gamma)\right]^2} - 1$, which lies entirely in the region $q > 1$. The parabolic stability boundary for $\varepsilon \ll \gamma_n^2$ ($\gamma_n \neq 0$) can also be obtained from a standing wave approximation to Eq. (1), in analogy to Milner's calculation [17]. This approximation, however, fails to reproduce Eq. (2) at larger values of ε.

Stability against transverse amplitude modulations can be studied by assuming that $\psi(x,y,t) = [1 + a(y,t)]\psi_0(x,q)$, with $a(y,t) = a_0(t)\cos(Qy)$ (Q small) and linearizing the resulting equation for $a(y,t)$. The TAM unstable region is given by,

$$\frac{3}{16}(q^2-1)^2 > \frac{3}{4}\gamma_n^2\alpha_q^4 + \gamma\gamma_n\alpha_q^2, \qquad (3)$$

and is shown in Fig. 1 as the shaded region. Instability occurs for finite Q. If $\gamma_n = 0$, all roll solutions are unstable to TAM. If, on the other hand, $\gamma_n \neq 0$, the region of wavenumbers around $q = 1$ is stable. At fixed γ, increasing γ_n increases the width of this stable region.

We next turn to the results of our numerical calculation. We use a pseudospectral method to solve Eq. (1) on a square grid of size $64\pi \times 64\pi$, with periodic boundary conditions. The number of Fourier modes used for each axis is 256. Time stepping is performed by a Crank-Nicholson scheme for the linear terms (including ψ^*), and a second order Adams-Bashforth scheme for the nonlinear terms. The time step used is $\Delta t = 0.1$. The initial condition, $\psi(x, y, t = 0)$ is a set of Gaussianly distributed random numbers, of zero mean and variance 10^{-4}. A typical run length is $10^5 - 10^6$ time steps.

Fourier modes with wavenumber close to $q = 1$ dominate in the early linear regime. As the system enters the nonlinear regime, the circularly averaged structure factor exhibits a dominant peak at $q = q_{max}$ which is seen to continuously shift away from 1 (detuning) and towards the Eckhaus stable region for all values of ε, i.e., the dominant wavenumber of the roll pattern shifts to $q > 1$. The value of q_{max} at long times is shown by the circles in Fig. (1). The error bars indicate the half-width at half-maximum of the peak. At $\varepsilon = \varepsilon_c(\gamma_n)$ the Eckhaus and TAM boundaries cross. For $\varepsilon < \varepsilon_c$, there is a region of stability against both Eckhaus and TAM perturbations. At sufficiently small ε, an asymptotic stationary roll state is reached with q_{max} inside this region. For sufficiently large values of ε, q_{max} enters the TAM unstable region first. Therefore the reentrant shape of the Eckhaus boundary and the fact that it crosses the TAM boundary at finite values of ε provides a mechanism for a finite threshold for the TAM instability. The shift in q_{max} and the position of the Eckhaus and TAM boundaries depend strongly on the value of γ_n.

We finally describe the asymptotic temporal dependence of the configurations as a function of ε. The results reported correspond to $\gamma = 0.1$ and $\gamma_n = 0.05$, but they are qualitatively similar in other cases provided that $\gamma_n \neq 0$ (for larger γ_n, both the appearance of the TAM and the transition to chaotic states are seen at larger ε). An almost perfect and stationary roll pattern is found for $\varepsilon = 0.02$ (Fig. 2(a)). As ε is increased to $\varepsilon = 0.05$ a very slowly varying transverse modulation of the roll pattern is observed (Fig. 2(b)). The wavelength of the modulation is about $3.2\lambda_0$, with $\lambda_0 = 2\pi$ the critical wavelength at onset. The periodic and almost stationary modulation suggests that the finite amplitude modulated

pattern at $\varepsilon = 0.05$ can be described by $\psi(x,y) = A_0(1 + b_0 \cos(Qy)) \cos qx + \cdots$. Fig. 3(b) shows the zeros of $\|\psi\|$ ($\|\psi\| < \|\psi\|_{max}/8$ is considered a zero). They all originate from the $\cos qx$ factor. Hence the TAM is weak enough that $\|1 + b_0 \cos(Qy)\| > 0$. At $\varepsilon = 0.1$ (Fig. 2(c)), the modulation becomes stronger and additional zeros of $\|\psi\|$ appear (Fig. 3(c)). We call the regions in which the slowly varying component of $\|\psi\|$ is zero a TAM defect. At this value of ε the wave pattern becomes time dependent, with the temporal variation of the wave pattern coming mostly from the motion of TAM defects. At $\varepsilon = 0.3$ (Fig. 2(d)), the length scale of the modulation becomes smaller, and the density of TAM defects increases (Fig. 3(d)). In order to characterize the temporal fluctuations of the wave patterns in this state, we have calculated the power spectrum of a time series of both ψ at a fixed point in space and the amplitude of a fixed Fourier mode. Figure 4 shows the time series of $Re[\psi]$ at a fixed point (Fig. 4(a)), and of the Fourier mode q=0.75 (Fig. 4(b)), and their corresponding power spectra (Fig. 4(c) and (d)) for $\varepsilon = 0.50$. The power spectra are broadband and decay as a power law of frequency ω^{-z} with $z \approx 4.0$ for frequencies in the range of $2 \times 10^{-3} < 2\omega/\Omega < 2 \times 10^{-2}$. Similar power law decay of the power spectra is also observed for $\varepsilon = 0.3$. The power law decay and the value of the exponent z are in good agreement with experimental results [5].

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FIGURES

FIG. 1. Stability diagram for $\gamma = 0.1$ and $\gamma_n = 0.05$ in the (q, ε) plane: boundary of existence of stationary roll solutions (dashed line, S); q_{max}, the peak of the circularly averaged structure factor calculated numerically (circles with error bars indicating the half-width at half-maximum of the peak); neutral stability curve (thin solid line, N); Eckhaus stability boundary (thick solid line, E); zig-zag stability boundary (long-dashed line, Z). The TAM unstable region is the shaded area.

FIG. 2. Typical configurations of $Re[\psi]$ at long times (shown in gray scale) for four values of ε. The initial configuration was random.

FIG. 3. Zeros of $\|\psi\|$ ($\|\psi\| < \|\psi\|_{max}/8$ is considered a zero) for four values of ε at the same times as in Fig. 2.

FIG. 4. (a) Time series of $Re[\psi]$ at a fixed point. (b) Time series of a Fourier mode $Re\left[\hat{\psi}(q_x = 0.75, q_y = 0)\right]$. (c) Power spectrum of the time series shown in (a). (d) Power spectrum of the time series shown in (b). The dashed line represents a power law with an exponent of -4.